Interpolant Strength

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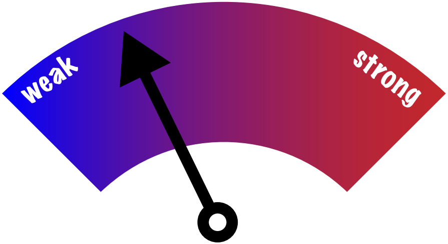
Verification with Model Checking And Interpolation

- Craig-interpolation commonly applied in model checking
  - used to compute approximate images
- Strongest interpolant not necessarily the best one
  - Coarse approximations can lead to faster convergence
- Range of interpolants exists
  - but existing interpolation systems can only generate one

Outline

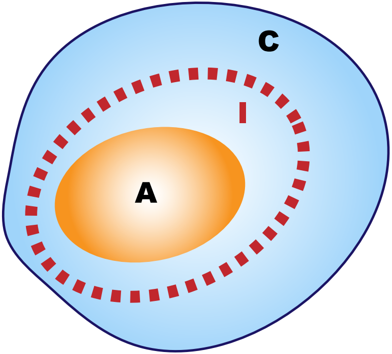
- Background
  - What is a Craig interpolant?
  - Interpolant-based model checking
  - Interpolation for propositional logic
- A novel, more general interpolation system
- Interpolant strength



What is a Craig interpolant?

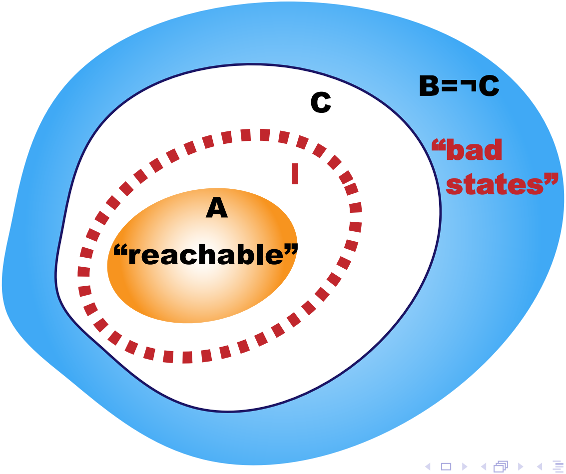
"Traditional" definition [William Craig, 57]:

- $A \Rightarrow I \Rightarrow C$
- all non-logical symbols in I occur in A *as well as* in C

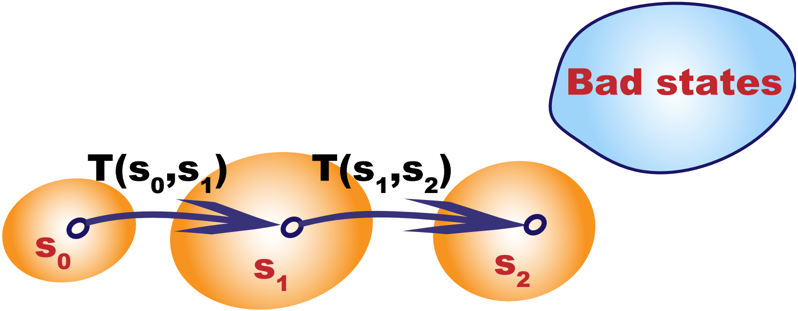


Common definition for automated verification:

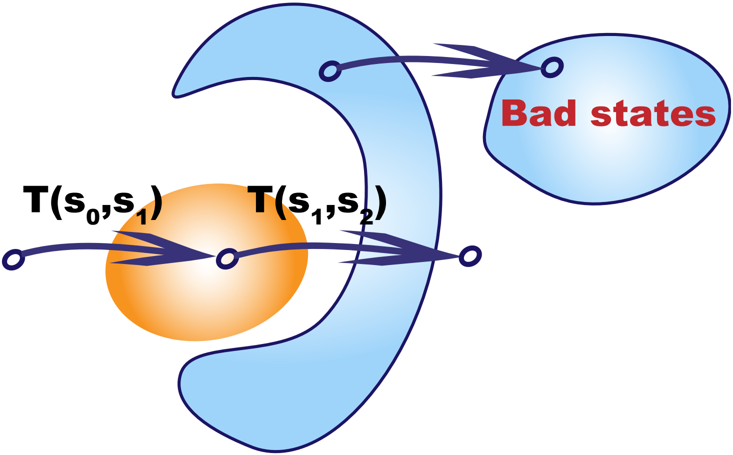
- $A \Rightarrow I$ and $I \wedge B$ inconsistent
- all non-logical symbols in I occur in A *as well as* in B

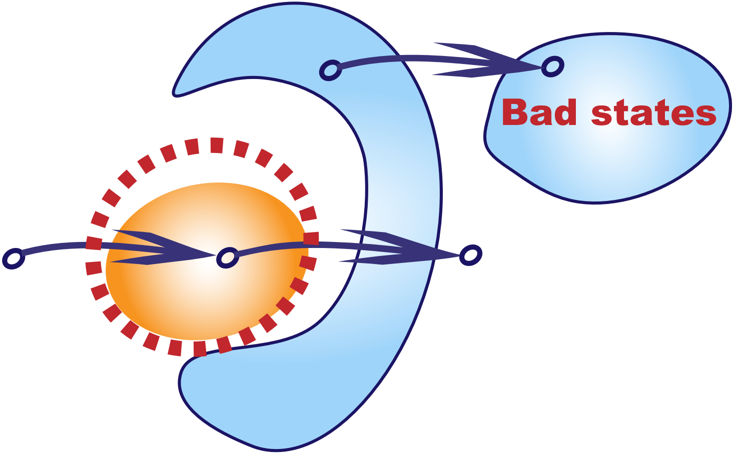


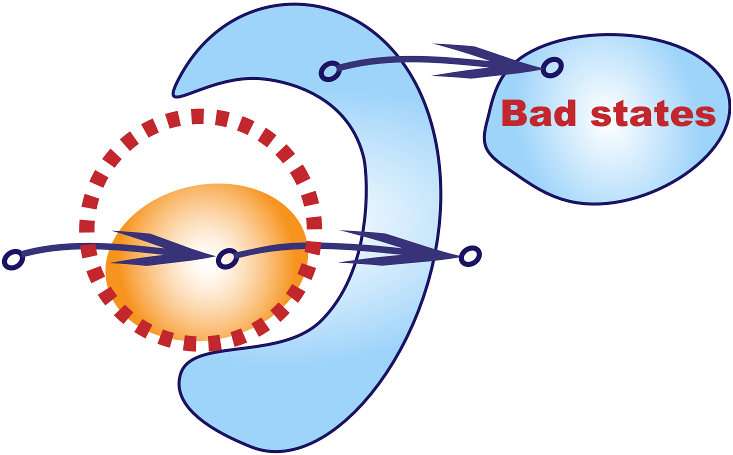
Bounded Model Checking

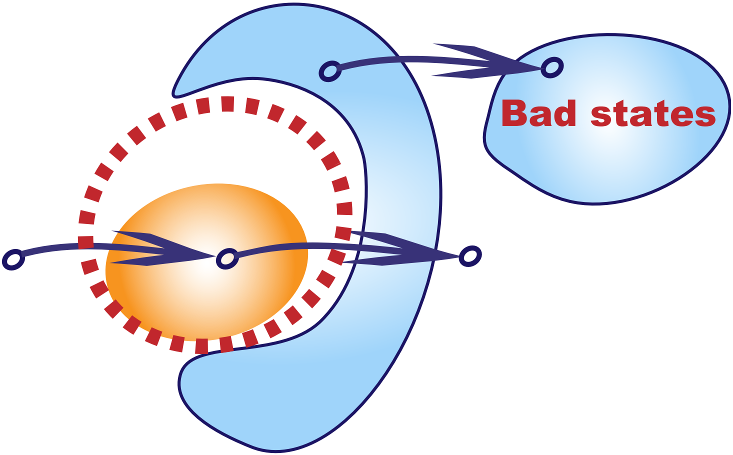


Interpolant-based Model Checking

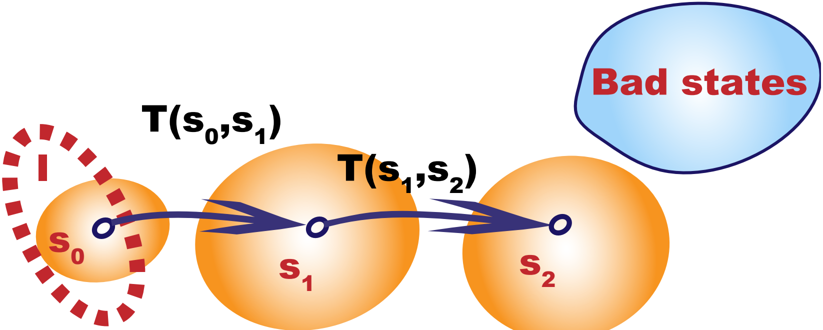



Bad states


Bad states


Bad states

Interpolant-based Model Checking: Iteration



- $x_0 = 0$
- $T(x_i, x_{i+1}) \equiv (x_{i+1} := x_i + 2)$
- Property: $x \neq 7$

$s_0 \cup I$	Strongest	Intermediate Interpolant	Weakest
$\{0\}$	$x_i = 2$	$x_i\%2 = 0$	$(x_i + 2) \neq 7$
$\{0, 2\}$	$x_i \in \{2, 4\}$	$x_i\%2 = 0$	$\ldots$
$\{0, 2, 4\}$	$x_i \in \{2, 4, 6\}$	$x_i\%2 = 0$	$\ldots$
$\{0, 2, 4, \ldots\}$	$x_i \in \{2, 4, 6, \ldots\}$	$x_i\%2 = 0$	$\ldots$

- Strongest interpolant delays convergence
- Weakest interpolant results in spurious counterexample

- CNF formula: A conjunction of clauses

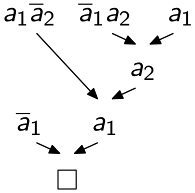
$$\bigwedge_i \bigvee_j \ell_{i,j}\,, \qquad \ell_{i,j} \in \{a, \overline{a} \mid a \in \text{Variables}\}$$

e.g.,

$$\overline{a}_1 \wedge (a_1 \vee \overline{a}_2) \wedge (\overline{a}_1 \vee a_2) \wedge a_1$$

- Resolution proofs

$$\frac{(C \vee a) \qquad (D \vee \overline{a})}{C \vee D} \quad \text{[Res]}$$



Resolution for CNF Formulas

- CNF formula: A conjunction of clauses

$$\bigwedge_i \bigvee_j \ell_{i,j}\,, \qquad \ell_{i,j} \in \{a, \overline{a} \mid a \in \text{Variables}\}$$

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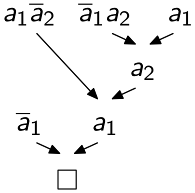
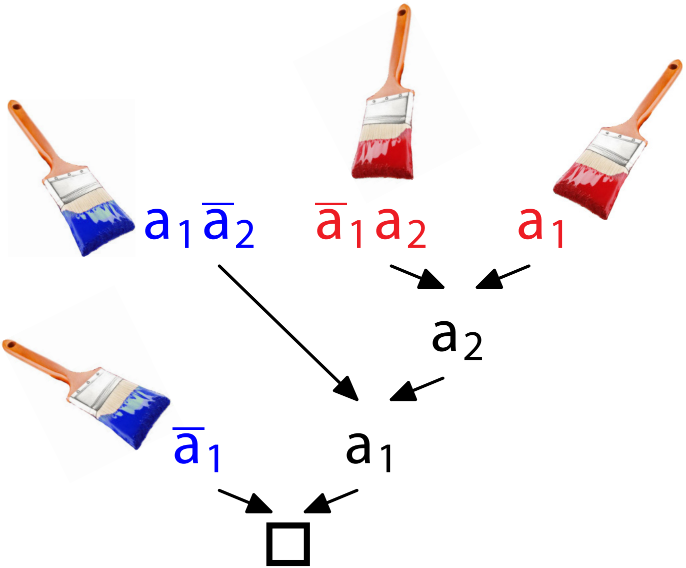


- Naturally generated by modern SAT solvers

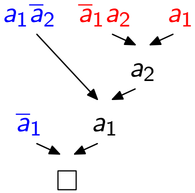
Colouring Formulas

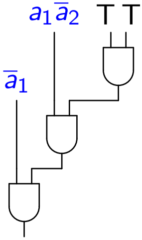
$A \equiv \overline{a}_1 \wedge (a_1 \vee \overline{a}_2)$ $B \equiv (\overline{a}_1 \vee a_2) \wedge a_1$

Interpolants from Resolution Proofs



- Interpolant I is a circuit following the structure of the proof
- In our example, I is
  - T if input values make $\overline{a}_1 \wedge (a_1 \vee \overline{a}_2)$ true
  - F if input values make $(\overline{a}_1 \vee a_2) \wedge a_1$ true





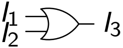
McMillan's Interpolation System

Annotate each clause C in the proof with a *partial interpolant* I

- Base case (initial clause C):
  -  $I =$ "keep all literals $\ell \in C$ s.t. $\mathrm{var}(\ell) \in \mathrm{Var}(B)$"
  -  $I = \mathrm{T}$
- Induction step (internal clauses C_1, C_2):

$$\frac{C_1 \vee a \quad [I_1] \qquad C_2 \vee \overline{a} \quad [I_2]}{C_1 \vee C_2 \quad [I_3]}$$

if $a \notin \mathrm{Var}(B)$, $\quad I_3 \stackrel{\text{def}}{=} I_1 \vee I_2$



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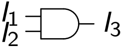
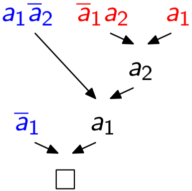
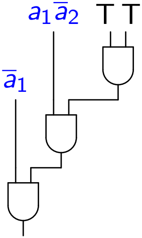


Interpolants from Proofs: Example Revisited





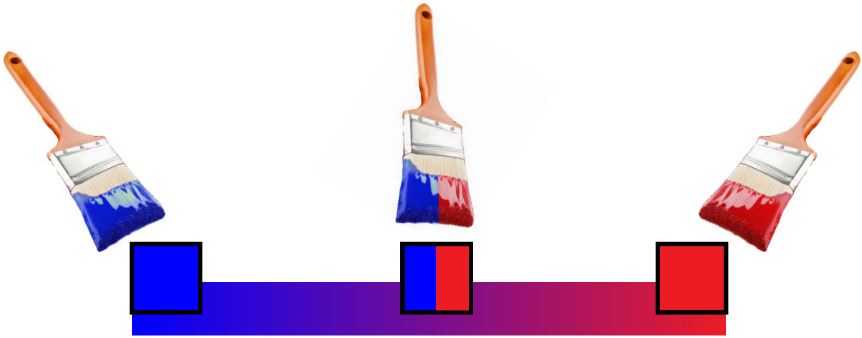
- I is $(\overline{a}_1 \wedge \overline{a_2})$, the strongest possible interpolant
- All interpolants form a lattice.

$$\overline{a}_1 \vee \overline{a}_2$$

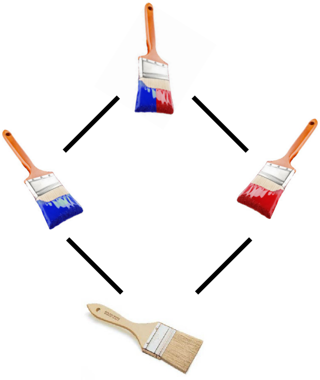
$$\overline{a}_1 \qquad\qquad \overline{a}_2$$

$$\overline{a}_1 \wedge \overline{a}_2$$

- Existing interpolation systems unnecessarily restrict *artistic freedom*



Lattice of Colours, Join

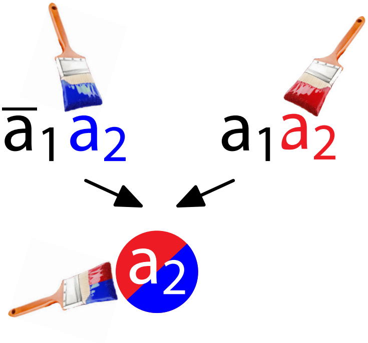


(colour lattice)

Locality Preserving Colouring

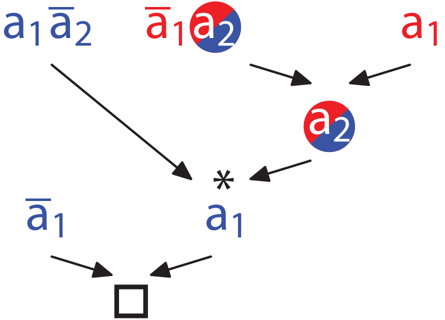
Each literal ℓ in each clause coloured separately!

- Literals from $A \setminus B$ must be coloured 
- Literals from $B \setminus A$ must be coloured 
- Literals from A and B: Any colour $\in$ {  ,  ,  }



$\bar{a}_1 a_2$
$a_1 a_2$
a_2

Example: Coloured Proof



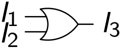
- $L(\overline{a}_2, u) \sqcup L(a_2, v) =$ 

Labelled Interpolation System

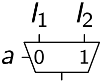
- Base case (initial vertices):
  - If $C \in$ A: $I \stackrel{\text{def}}{=}$ “all literals $\ell \in C$ s.t. $L(\ell, v) =$  ”
  - If $C \in$ B: $I \stackrel{\text{def}}{=} \neg($ “all literals $\ell \in C$ s.t. $L(\ell, v) =$  ”)
- Induction step (internal vertices):

$$\frac{C_1 \vee a \quad [I_1] \qquad C_2 \vee \bar{a} \quad [I_2]}{C_1 \vee C_2 \quad [I_3]}$$

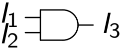
if $L(a) \sqcup L(\bar{a}) =$  $I_3 \stackrel{\text{def}}{=} I_1 \vee I_2$



if $L(a) \sqcup L(\bar{a}) =$  $I_3 \stackrel{\text{def}}{=} (a \vee I_1) \wedge (I_2 \vee \bar{a})$



if $L(a) \sqcup L(\bar{a}) =$  $I_3 \stackrel{\text{def}}{=} I_1 \wedge I_2$



Theorem. *For any* (A, B)*-refutation* R *and locality preserving colouring* L*,* $\text{ITP}(L, R)$ *is an interpolant for* (A, B)*.*

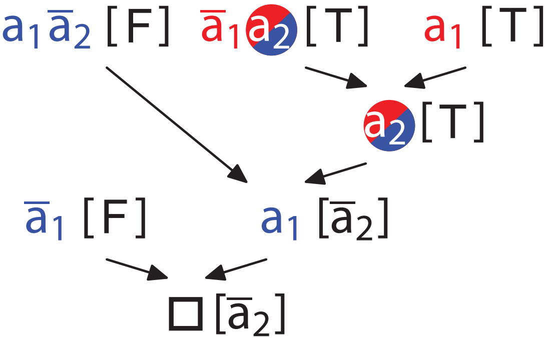
Proof: Minor adaptation of [Yorsh and Musuvathi, CADE '05]:

Invariant:

$$A \wedge (\neg C|_A) \vdash I$$

$$B \wedge (\neg C|_B) \vdash \neg I$$

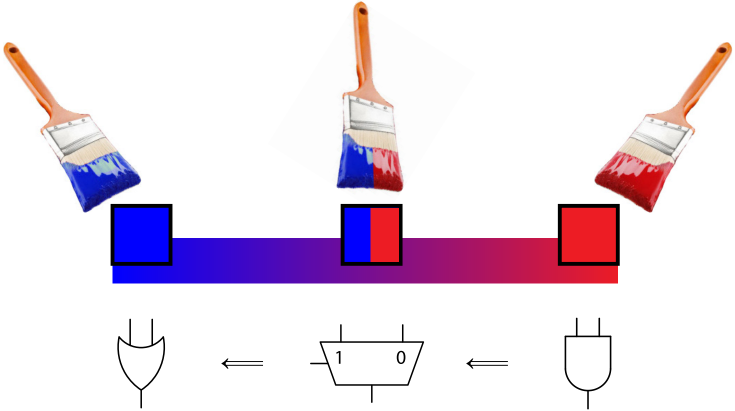
$$\mathrm{Var}(I) \subseteq \mathrm{Var}(A) \cap \mathrm{Var}(B)$$



- Interpolant $\overline{a_2}$ cannot be obtained with existing systems!
- Also, $\overline{a_2}$ is implied by $\overline{a}_1 \wedge \overline{a}_2$.

Strength of Interpolants

$$I_1 \vee I_2 \Longleftarrow (a \vee I_1) \wedge (I_2 \vee \overline{a}) \Longleftarrow I_1 \wedge I_2$$



Theorem: Strength of Interpolants

- min and max for colours
- Lift $\Leftarrow$, min and max pointwise to colouring functions

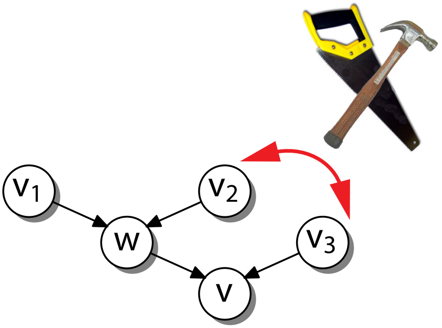
Theorem. *Let R be an* (A, B)*-refutation and* $\mathbb{L}_R$ *be the set of locality preserving colourings over R. The structure* $(\mathbb{L}_R, \Leftarrow, \max, \min)$ *is a complete lattice.*

Strength of Interpolants (continued)

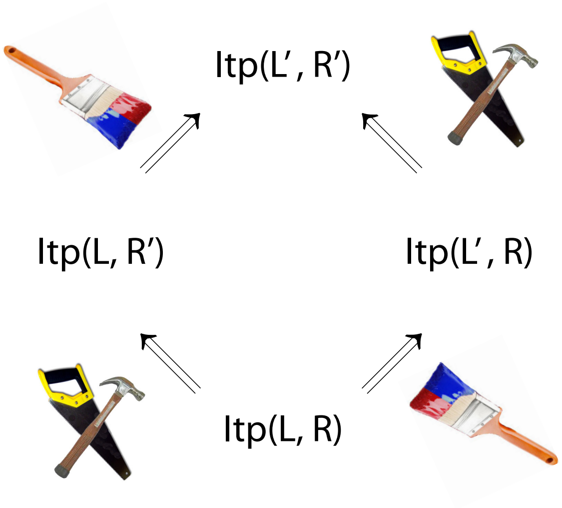
	A-local	*A*/*B*-shared	*B*-local	
strongest				(McMillan)
		 . . . 		
⇓				(Huang, Krajíček, Pudlák)
		 . . . 		
weakest				(“inverse” McMillan)

Restructuring Proofs

- Change strength of interpolant by swapping nodes in proof
- Informally introduced in [Jhala and McMillan, LMCS 07]



- Labelling and restructuring are orthogonal techniques!



- Labelled interpolation systems
  - generalise existing interpolation systems for propositional logic
  - constitute a dial for tuning interpolant strength
- All proofs available in ETH Technical Report 652
- Vijay D'Silva, ESOP 2010:

  Propositional Interpolation and
  Abstract Interpretation

  - Interpolation systems, clauses and interpolants form abstract domains.
  - Existing systems as optimal abstractions of the colouring system.
- Future work
  - Empirical analysis of effect of interpolant strength